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AN ELASTIC–PLASTIC ANALYSIS OF AN ASYMMETRIC CRACKED SPECIMEN SUBJECTED TO LONGITUDINAL SHEAR

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Abstract—Some recent elastic–plastic analyses of cracked specimens subjected to symmetric mode III loading are extended to include asymmetric loading and geometry. Solutions are given for arbitrary work hardening behaviour in any specimen that is amenable to a linear elastic analysis. It is shown that asymmetry has a major influence on the shape of the plastic zone, but does not affect the J-integral unil the loading is well into the large scale yielding range. In particular the "plastic zone corrected" estimate of J, obtained by elastically solving a problem for a crack longer than the actual one, is shown to remain a valid two-term asymptotic expansion in the presence of asymmetry. The general results are applied to a crack at an angle to a uniform stress field in a power law hardening material. The growth of the plastic zone is displayed graphically for various hardening exponents and crack orientations. No other asymmetric solution is available, but values of J are compared with those obtained from a fully plastic analysis in the symmetric case.

1. INTRODUCTION

Two recent papers have applied the method of matched asymptotic expansions to elastic–plastic crack problems. Edmunds and Willis [1] generated solutions applicable to the longitudinal shear loading of an elastic–perfectly plastic symmetric specimen. This analysis was extended by Edmunds and Willis [2] to allow arbitrary work-hardening behaviour. The current paper further extends the analysis to include problems in which the loading and geometry are not symmetric about the crack plane. The previous results are special cases of those given here.

The method involves finding two expansions that are asymptotic to the required solution. Both are power series expansions in a monotonically increasing loading parameter but they have different regions of validity. The first, developed in Section 2, is valid in an "outer" region away from the crack tip and the second, developed in Section 3, in an "inner" region near the tip. Unknown constants that appear in each expansion are determined in Section 4 by an application of Van Dyke's [3] "asymptotic matching principle". The discussion of the solution, Section 5, concentrates on its application to fracture mechanics; it being shown how asymptotic expressions for parameters such as the extent of yielding or the J-integral can be found from elastic solutions alone. To lowest order these expressions are the well known small scale yielding results, the range of validity of which are extended by the higher order terms. These general expressions are applied in Section 6 to the particular problem of a crack at an angle to a uniform stress field in a power law hardening material. The growth of the plastic zone with the loading parameter is displayed graphically for various crack orientations and levels of hardening. For the particular orientation of the crack parallel to the plane of maximum applied shear the results are compared with a fully plastic analysis. The dependence of various estimates of J on the applied shear is displayed for two levels of hardening.

The formulation of the governing equations is identical to that given in [2], and also in Rice [4], and so will only be summarised here. For the longitudinal shear deformation of the specimen shown in Fig. 1 the only non-zero component of displacement is $U_3(x_1, x_2)$, and hence the only non-zero stresses are σ_{13} and σ_{23}. In this case the general constitutive relation is

$$\sigma_{i3} = \frac{\tau(\gamma)}{\gamma}\frac{\partial U_3}{\partial x_i} = \frac{\tau(\gamma)}{\gamma}\gamma_{i3}, \quad i = 1, 2 \tag{1.1}$$

where $\tau(\gamma)$ is a given function relating the principal shear stress $\tau = (\sigma_{13}^2 + \sigma_{23}^2)^{1/2}$ to the principal shear strain $\gamma = (\gamma_{13}^2 + \gamma_{23}^2)^{1/2}$. If the material is elastic–plastic the function $\tau(\gamma)$ satisfies

$$\tau(\gamma) = \frac{k}{\gamma_0}\gamma \quad \text{for } \gamma \leq \gamma_0 \tag{1.2}$$

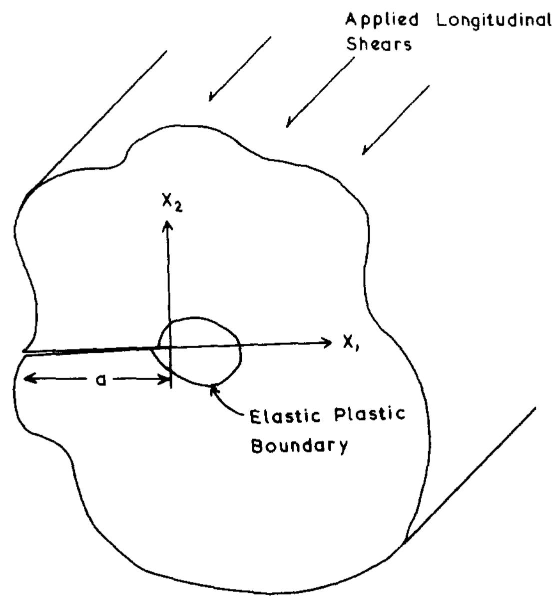

Fig. 1. The specimen geometry and loading.

where k and γ_0 are the initial yield stress and strain respectively. It will be seen that (1.1) defines a "deformation" theory of plasticity which, it is hoped, is a reasonable approximation to an "incremental" theory when the loading is monotonic. Apart from the constitutive law (1.1), the solution must satisfy the equilibrium condition

$$\frac{\partial \sigma_{13}}{\partial x_1}+\frac{\partial \sigma_{23}}{\partial x_2}=0 \tag{1.3}$$

and be compatible with the prescribed boundary tractions.

In the elastic region, (1.2) and (1.3) can be satisfied by requiring U_3 to be a harmonic function of x_1, x_2. This can be conveniently expressed as

$$U_3(x_1, x_2)=a\gamma_0\, Im\,\{g(z)\} \tag{1.4}$$

where a is the crack length, as in Fig. 1, and $g(z)$ is an analytic function of

$$z=\frac{x_1+ix_2}{a}. \tag{1.5}$$

In addition to this complex position variable it is convenient to define a complex strain variable by

$$\xi=\frac{\gamma_{23}-i\gamma_{13}}{\gamma_0}. \tag{1.6}$$

Application of the Cauchy–Riemann equations then shows that

$$\bar{\xi}=g'(z), \tag{1.7}$$

which can be inverted to

$$\bar{z}=f'(\xi),\quad \text{say}. \tag{1.8}$$

In the strain plane, Fig. 2, the traction free crack faces map onto the $\gamma_{23}=0$ axis, the elastic–plastic boundary onto a semi-circle of radius γ_0 and the outer boundary of the specimen onto α, possibly complex, curve surrounding the origin.

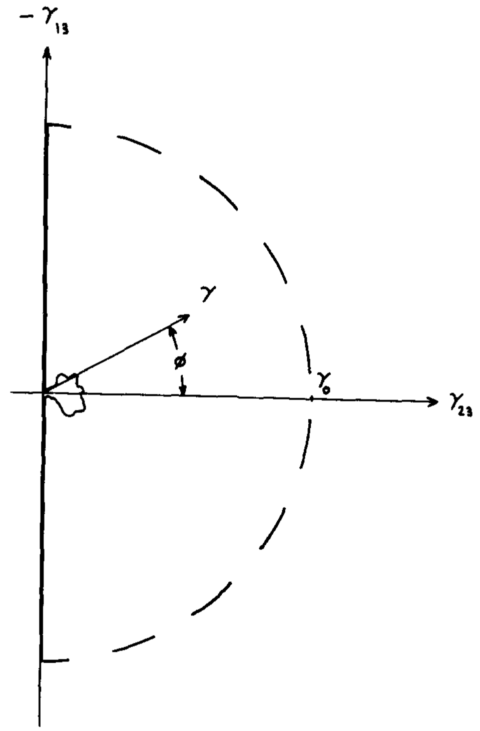

Fig. 2. The map of the specimen onto the strain plane.

2. THE OUTER EXPANSION

As in [1] and [2] the outer expansion is found by formulating and solving a boundary value problem for $g(z)$. The real part of $g(z)$ on the specimen boundary follows from the prescribed tractions and, as before, the yielded zone is modelled by a crack tip singularity. The difference between this solution and that given in [1] and [2] is that the symmetry condition

$$U_3(x_1, x_2) = -U_3(x_1, -x_2) \tag{2.1}$$

is no longer applicable. This means that the singularity can include negative integral powers of z, as well as the negative half integral powers previously used. A sufficiently general form is thus

$$\begin{aligned} g(z) \sim -i(\alpha_1\epsilon^3 + \alpha_2\epsilon^5 + 0(\epsilon^7)) \log(-z) - 2(\alpha_3\epsilon^3 + \alpha_4\epsilon^5 + 0(\epsilon^7))z^{-1/2} \\ - i(\alpha_5\epsilon^5 + 0(\epsilon^7))z^{-1} - \tfrac{2}{3}(\alpha_6\epsilon^5 + 0(\epsilon^7))z^{-3/2} + 0(\epsilon^7)z^{-2} \quad \text{as } z \to 0 \end{aligned} \tag{2.2}$$

where the α_i are, as yet, undetermined and the loading parameter ϵ is defined as the ratio of the maximum applied boundary stress to the yield stress, k.

Modelling the yielded zone by (2.2) leads to a boundary value problem for $g(z)$ which can be solved and differentiated to give the three term outer asymptotic expansion. This has the form

$$\bar{\xi} = g'(z) = g_1'(z)\epsilon + ig_2'(z)(\alpha_1\epsilon^3 + \alpha_2\epsilon^5) + g_3'(z)(\alpha_3\epsilon^3 + \alpha_4\epsilon^5) + ig_4'(z)\alpha_5\epsilon^5 + g_5'(z)\alpha_6\epsilon^5 + 0(\epsilon^7) \tag{2.3}$$

where:

$g_1'(z)\epsilon$ is the linear elastic solution of the problem

$g_2'(z)$ has a unit z^{-1} stress singularity

$g_3'(z)$ has a unit $z^{-3/2}$ stress singularity

$g_4'(z)$ has a unit z^{-2} stress singularity

$g_5'(z)$ has a unit $z^{-5/2}$ stress singularity, and $ig_2'(z)$, $g_3'(z)$, $ig_4'(z)$ and $g_5'(z)$ are solutions giving zero boundary tractions.

As the inner limit of (2.3) will be used in the subsequent matching the small argument expansions of $g_i'(z)$ are introduced as

$$\begin{aligned} g_1'(z) &= \gamma_{11}z^{-1/2} + i\gamma_{12} + \gamma_{13}z^{1/2} + i\gamma_{14}z + \gamma_{15}z^{3/2}\ldots\ldots \\ g_2'(z) &= z^{-1} + i\gamma_{22}z^{-1/2} + \gamma_{23} + i\gamma_{24}z^{1/2} + \gamma_{25}z\ldots\ldots \\ g_3'(z) &= z^{-3/2} + \gamma_{33}z^{-1/2} + i\gamma_{34} + \gamma_{35}z^{1/2}\ldots\ldots \\ g_4'(z) &= z^{-2} + i\gamma_{44}z^{-1/2} + \gamma_{45}\ldots\ldots \\ g_5'(z) &= z^{-5/2} + \gamma_{55}z^{-1/2}\ldots\ldots \quad \text{for } z \to 0. \end{aligned} \tag{2.4}$$

It may be noted that, unlike the equivalent functions in [1] these $g_i'(z)$ are given uniquely by the solutions of elastic problems.

3. THE INNER EXPANSION

As in [2] the inner expansion is found by formulating and solving a boundary value problem for the function $f'(\xi)$, defined by (1.8). The elastic region is represented by the area within the dashed semi-circle in Fig. 2, and the curve representing the specimen boundary lies within a distance of order $\gamma_0\epsilon$ of the origin, since the boundary conditions imply $\xi = 0(\epsilon)$ there. The function $f'(\xi)$ is consequently taken as analytic within the semi-circle, except at the origin where a singularity is admitted to model the specimen boundary strains. The functional form of this singularity is found by noting that it cannot contain terms such as $\epsilon^3\xi^2$ or $i\epsilon^4\xi^3$, since ξ is an odd function of ϵ, and that terms in ξ^{-1} are incompatible with the plastic region solution given below. These considerations, together with the boundary condition $Im\ \{f'(\xi)\} = 0$ when $Re\ \{\xi\} = 0$, lead to the elastic region solution

$$\bar{z} = f'(\xi) = (\beta_1\epsilon^2 + \beta_2\epsilon^4 + \beta_3\epsilon^6)\xi^{-2} + i(\beta_4\epsilon^3 + \beta_5\epsilon^5)\xi^{-3} + (\beta_6\epsilon^4 + \beta_7\epsilon^6)\xi^{-4} + i\beta_8\epsilon^5\xi^{-5} + \beta_9\epsilon^6\xi^{-6} + F'(\xi) + 0(\epsilon^7) \tag{3.1}$$

where the β_i are, as yet, undetermined. As in [2] the function $F'(\xi)$ is analytic in the unit semi-circle and is determined from conditions on the elastic–plastic boundary.

The determination of $F'(\xi)$ follows [2], and hence Rice [4], in that a potential function $\psi(\gamma_{13}, \gamma_{23})$ is introduced such that the plastic region co-ordinates and displacements are

$$x_i = \frac{\partial\psi}{\partial\gamma_{i3}}, \quad i = 1, 2 \tag{3.2}$$

and

$$U_3(x_1, x_2) = \gamma_{i3}x_i - \psi + \text{const.} \tag{3.3}$$

respectively. This formulation automatically satisfies compatibility and the equilibrium condition becomes [4]

$$\frac{\tau(\gamma)}{\gamma\tau'(\gamma)}\frac{\partial^2\psi}{\partial\gamma^2} + \frac{1}{\gamma}\frac{\partial\psi}{\partial\gamma} + \frac{1}{\gamma^2}\frac{\partial^2\psi}{\partial\phi^2} = 0 \tag{3.4}$$

where (γ, ϕ) are polar co-ordinates given by

$$\gamma_{23} - i\gamma_{13} = \gamma\, e^{i\phi} \tag{3.5}$$

and $\tau(\gamma)$ is the constitutive relation defined in (1.1). The crack surfaces, where $x_2 = 0$ and $\gamma_{23} = 0$, correspond to $\phi = \pm(\pi/2)$ and hence, from (3.2),

$$\frac{\partial\psi}{\partial\phi} = 0 \quad \text{when } \phi = \pm\frac{\pi}{2}. \tag{3.6}$$

Solutions of (3.4), subject to (3.6), are now developed by separation of variables to give

$$\psi = \sum_{k=1}^{\infty} D_k f_k(\gamma)\sin(2k-1)\phi + \sum_{k=1}^{\infty} D_k^* f_k^*(\gamma)\cos 2k\phi. \tag{3.7}$$

It will be noted that, unlike the solution given in [2] and [4], cosine terms are included in (3.7). Substituting (3.7) into (3.4) shows that $f_k(\gamma)$ and $f_k^*(\gamma)$ satisfy

$$\frac{\tau(\gamma)}{\gamma\tau'(\gamma)} f_k''(\gamma) + \frac{1}{\gamma} f_k'(\gamma) - (2k-1)^2 \frac{1}{\gamma^2} f_k(\gamma) = 0 \tag{3.8}$$

and

$$\frac{\tau(\gamma)}{\gamma\tau'(\gamma)} f_k^{*\prime\prime}(\gamma) + \frac{1}{\gamma} f_k^{*\prime}(\gamma) - 4k^2 \frac{1}{\gamma^2} f_k^*(\gamma) = 0 \tag{3.9}$$

respectively. The conditions

$$f_k(\gamma_0) = f_k^*(\gamma_0) = 1 \quad \text{and} \quad f_k'(\infty) = f_k^{*\prime}(\infty) = 0 \tag{3.10}$$

are imposed on (3.8) and (3.9) to normalise the functions and ensure that the strain singularity lies at the crack tip.

The co-ordinates of the elastic-plastic boundary are found by substituting (3.7) into (3.2) and setting $\gamma = \gamma_0 e^{i\phi}$. This gives

$$x_1 - ix_2 = \frac{-e^{-i\phi}}{2\gamma_0}\Big(\sum_{k=1}^{\infty} D_k\{[2k-1+\gamma_0 f_k'(\gamma_0)]\,e^{i(2k-1)\phi} + [2k-1-\gamma_0 f_k'(\gamma_0)]\,e^{-i(2k-1)\phi}\}$$
$$+ i\sum_{k=1}^{\infty} D_k^*\{[2k+\gamma_0 f_k^{*\prime}(\gamma_0)]\,e^{i2k\phi} - [2k-\gamma_0 f_k^{*\prime}(\gamma_0)]\,e^{-i2k\phi}\}\Big). \tag{3.11}$$

A second expression for these co-ordinates is obtained from (3.1) by performing a power series expansion of $F'(\xi)$ and setting $\xi = e^{i\phi}$. The two expressions have coefficients of $e^{ik\phi}$ equated and, as in [2], the D's and $F'(\xi)$ are determined to a certain order in ϵ. Substituting the result into (2.1) gives

$$\bar{z} = (\beta_1\xi^{-2} + \beta_1 C_1)\epsilon^2 + i(\beta_4\xi^{-3} - \beta_4 C_1^*\xi)\epsilon^3 + (\beta_2\xi^{-2} + \beta_6\xi^{-4} + \beta_2 C_1 + \beta_6 C_2\xi^2)\epsilon^4$$
$$+ i(\beta_5\xi^{-3} + \beta_8\xi^{-5} - \beta_5 C_1^*\xi - \beta_8 C_2^*\xi^3)\epsilon^5$$
$$+ (\beta_3\xi^{-2} + \beta_7\xi^{-4} + \beta_9\xi^{-6} + \beta_3 C_1 + \beta_7 C_2\xi^2 + \beta_9 C_3\xi^4)\epsilon^6 + 0(\epsilon^7) \tag{3.12}$$

with

$$C_k = (2k-1+\gamma_0 f_k'(\gamma_0))/(2k-1-\gamma_0 f_k'(\gamma_0))$$

and

$$C_k^* = (2k+\gamma_0 f_k^{*\prime}(\gamma_0))/(2k-\gamma_0 f_k^{*\prime}(\gamma_0)). \tag{3.13}$$

Equation (3.12) can be iteratively inverted to give the required expansion of $\bar{\xi}(z)$ that is valid in the inner region. The resulting expression is lengthy and will not be given explicitly.

4. MATCHING

So far two expansions that are asymptotic to the unknown solution have been obtained. The outer expansion (2.3), gives the first three terms of $\bar{\xi}(z)$ and is valid when $z \sim 1$ and the inner expansion (3.12), gives the first five terms of $\bar{z}(\xi)$ when $z \sim \epsilon^2$. These are matched by first iteratively inverting (3.12) to give five terms of $\bar{\xi}(z)$ and then, as in [1] and [2], equating the

m-term outer expansion of the n-term inner expansion to the n-term inner expansion of the m-term outer expansion. In this case $m = 3$ and $n = 5$ and, after lengthy algebra, there results:

$$\beta_1 = \gamma_{11}^2$$

$$\beta_4 = -2\gamma_{12}\gamma_{11}^2$$

$$\beta_6 = 2\gamma_{13}\gamma_{11}^3 - 3\gamma_{12}^2\gamma_{11}^2$$

$$\beta_8 = 4\gamma_{12}^3\gamma_{11}^2 - 8\gamma_{12}\gamma_{13}\gamma_{11}^3 - 2\gamma_{14}\gamma_{11}^4$$

$$\beta_9 = 2\gamma_{15}\gamma_{11}^5 + 5\gamma_{12}^4\gamma_{11}^2 - 20\gamma_{13}\gamma_{12}^2\gamma_{11}^3 - 10\gamma_{12}\gamma_{14}\gamma_{11}^4 + 5\gamma_{13}^2\gamma_{11}^4$$

$$\alpha_3 = \tfrac{1}{2}C_1\gamma_{11}^3$$

$$\alpha_1 = 0$$

$$\beta_2 = C_1(\gamma_{33}\gamma_{11}^4 + \gamma_{13}\gamma_{11}^3)$$

$$\beta_5 = -C_1(2\gamma_{12}\gamma_{33}\gamma_{11}^4 + 2\gamma_{12}\gamma_{13}\gamma_{11}^3 + 2\gamma_{14}\gamma_{11}^4 + \gamma_{34}\gamma_{11}^5)$$

$$\beta_7 = C_1(\gamma_{35}\gamma_{11}^6 + 3\gamma_{15}\gamma_{11}^5 - 3\gamma_{12}^2\gamma_{33}\gamma_{11}^4 - 3\gamma_{12}^2\gamma_{13}\gamma_{11}^3 + 3\gamma_{13}\gamma_{33}\gamma_{11}^5 + 3\gamma_{13}^2\gamma_{11}^4 - 6\gamma_{12}\gamma_{14}\gamma_{11}^4 - 3\gamma_{12}\gamma_{34}\gamma_{11}^5)$$

$$\alpha_6 = \tfrac{3}{8}C_1{}^2\gamma_{11}^5$$

$$\alpha_5 = -C_1^*\gamma_{12}\gamma_{11}^4$$

$$\alpha_4 = C_1^*\gamma_{12}^2\gamma_{11}^3 + \tfrac{1}{8}C_1{}^2(6\gamma_{33}\gamma_{11}^5 + 5\gamma_{13}\gamma_{11}^4)$$

$$\alpha_2 = 0$$

$$\beta_3 = C_1^*(2\gamma_{12}^2\gamma_{33}\gamma_{11}^4 + 2\gamma_{12}\gamma_{44}\gamma_{11}^5 + 2\gamma_{12}^2\gamma_{13}\gamma_{11}^3 + 2\gamma_{12}\gamma_{14}\gamma_{11}^4) + \tfrac{1}{4}C_1{}^2(7\gamma_{33}^2\gamma_{11}^6 + 11\gamma_{13}\gamma_{33}\gamma_{11}^5 + 5\gamma_{13}^2\gamma_{11}^4 + 3\gamma_{55}\gamma_{11}^6 + 2\gamma_{35}\gamma_{11}^6 + 3\gamma_{15}\gamma_{11}^5). \quad (4.1)$$

Equation (4.1) gives the previously undetermined singularity coefficients, the α_i and β_i, in terms of γ_{ij}, C_1 and C_1^*. The γ_{ij} are defined from linear elastic solutions for the specimen and C_1 and C_1^* are given, in principle, by solving (3.8) and (3.9) with the appropriate constitutive relation. The values of α_i and β_i thus found can be substituted into (2.3) and (3.12) to give asymptotic solutions valid in the outer and inner regions. If the intermediate region were of interest a uniformly valid composite expansion could be constructed by the methods discussed in [3].

5. APPLICATIONS TO FRACTURE MECHANICS

As noted in the introduction the discussion of the solution completed in Section 4 concentrates on applications to fracture mechanics. In particular the solution may be characterised by the size and shape of the yielded zone and the path independent energy integral J. The first of these is included because the yielded zones display an interesting asymmetry and the second because of recent interest, e.g. Begley and Landes [5], in J as a general yielding fracture criteria. General expressions for these are developed and the case of a power law hardening material is considered in detail, it being shown how the crack tip strain singularity is essentially unaltered by the presence of asymmetry.

The co-ordinates of the elastic–plastic boundary are found from (3.12) with β_i given by (4.1) and $\xi = e^{i\phi}$. The resulting five term expansion is lengthy and so only the first three terms are given here. Thus, to three terms,

$$\bar{z}|_{E.P.} = \gamma_{11}^2(e^{-2i\phi} + C_1)\epsilon^2 + 2i\gamma_{12}\gamma_{11}^2(C_1^* e^{i\phi} - e^{-3i\phi})\epsilon^3 + ((2\gamma_{13}\gamma_{11}^3 - 3\gamma_{12}^2\gamma_{11}^2)(e^{-4i\phi} + C_2 e^{2i\phi}) + C_1(\gamma_{33}\gamma_{11}^4 + \gamma_{13}\gamma_{11}^3)(e^{-2i\phi} + C_1))\epsilon^4 + 0(\epsilon^5). \quad (5.1)$$

The maximum extent of this zone is found by equating the derivative of $|z|$ with respect to ϕ to zero. The resulting value is

$$|z|_{max} = \gamma_{11}^2(1+c_1)\epsilon^2 + \left[\frac{\gamma_{12}^2\gamma_{11}^2}{2C_1(1+C_1)}(3C_1^* + 1 + C_1C_1^* + 3C_1)^2 + \frac{2\gamma_{12}^2\gamma_{11}^2}{(1+C_1)}(C_1^* - 1)^2\right.$$
$$\left. + ((2\gamma_{13}\gamma_{11}^3 - 3\gamma_{12}^2\gamma_{11}^2)(1+C_2) + C_1(1+C_1)(\gamma_{33}\gamma_{11}^4 + \gamma_{13}\gamma_{11}^3))\right]\epsilon^4 + 0(\epsilon^6) \tag{5.2}$$

which occurs when

$$\phi = -\frac{\gamma_{12}}{2C_1}(3C_1^* + 1 + C_1C_1^* + 3C_1)\epsilon + 0(\epsilon^2). \tag{5.3}$$

The first terms of (5.1) and (5.2) are the well known small scale yielding solutions associated with a stress intensity factor of $K_{III} = (2\pi a)^{1/2}\gamma_{11}k\epsilon$.

As in [2] the path independent energy integral J is calculated from the outer solution (2.3). The expression used is

$$J = Im\left\{\frac{k\gamma_0 a}{2}\oint_\Gamma \bar{\xi}^2\,dz\right\}, \tag{5.4}$$

given by Rice and Budiansky[6], and the contour Γ is taken as the circle

$$z = \frac{\rho}{a}e^{i\theta}. \tag{5.5}$$

Performing the substitutions is is found that only the z^{-1} term of $\bar{\xi}^2$ contributes to the integral and hence J can be evaluated in terms of the γ_{ij} defined in (2.4). The result is

$$J = \pi ak\gamma_0\left[\gamma_{11}^2\epsilon^2 + c_1(\gamma_{33}\gamma_{11}^4 + \gamma_{13}\gamma_{11}^3)\epsilon^4\right.$$
$$+ \left(\frac{C_1^2}{4}(7\gamma_{33}^2\gamma_{11}^6 + 11\gamma_{13}\gamma_{33}\gamma_{11}^5 + 5\gamma_{13}^2\gamma_{11}^4 + 3\gamma_{55}\gamma_{11}^6 + 2\gamma_{35}\gamma_{11}^6 + 3\gamma_{15}\gamma_{11}^5)\right.$$
$$\left.\left. + C_1^*(2\gamma_{12}^2\gamma_{33}\gamma_{11}^4 + 2\gamma_{12}\gamma_{44}\gamma_{11}^5 + 2\gamma_{12}^2\gamma_{13}\gamma_{11}^3 + 2\gamma_{12}\gamma_{14}\gamma_{11}^4)\right)\epsilon^6 + 0(\epsilon^8)\right]. \tag{5.6}$$

As before, the first term of (5.6),

$$J = \pi ak\gamma_0\gamma_{11}^2\epsilon^2, \tag{5.7}$$

is the small yielding result and the second and third are systematic refinements that take account of the yielding.

The above expressions can be used to investigate the effect of asymmetry on fracture criteria. It can be seen immediately that the small scale yielding approximations depend only on γ_{11}, and hence are unaffected by asymmetry which involves the coefficients γ_{12}, γ_{14}, etc. (5.1) shows that these asymmetric coefficients affect the elastic-plastic boundary co-ordinates by a term of order ϵ^3, but (5.2) shows that the effect on the maximum extent of yielding is of order ϵ^4. These comparatively strong effects are in contrast with those for the J integral, (5.6), where asymmetry has no influence until the term of order ϵ^6. It was shown in [2] that a "plastic zone correction" obtained by solving a linear elastic problem for a crack of length $a + r_y$, r_y being given by

$$r_y = aC_1\gamma_{11}^2\epsilon^2, \tag{5.8}$$

provides a two term asymptotic expansion for J. This result can be seen to remain valid in the presence of asymmetry since neither r_y nor the first two terms of (5.6) are affected. Finally, it may be noted from (5.1) that the values of $\bar{z}|_{E.P.}$ are not equal on the two crack faces, i.e. when $\phi = \pm(\pi/2)$. This means that the analogue of crack opening displacement used in [2], where the relative displacement of the plastic zone ends was considered, ceases to be meaningful.

It is also of interest to compare the general expressions (5.1) and (5.6) with those obtained from a modified boundary layer approach. In this the problem is first solved elastically to find γ_{11} and γ_{12}, and then a semi-infinite crack in an infinite body is considered. This is subjected to the remote boundary condition

$$\bar{\xi} \to \gamma_{11} z^{-1/2}\epsilon + i\gamma_{12}\epsilon \quad \text{as } z \to \infty \tag{5.9}$$

and an elastic–plastic calculation is performed. The resulting plastic zone and J-integral are taken as approximations to those that would be found in an elastic–plastic solution of the original problem. For general hardening behaviour the boundary layer problem can be solved, up to a certain order in ϵ, by a method similar to that of Section 3. As expected the results are given by (5.1) and (5.6), except that now γ_{11} and γ_{12} are the only non-zero γ_{ij}. This means that the modified boundary layer approach gives the correct ϵ^3 term in the elastic–plastic boundary co-ordinates, but the incorrect ϵ^4 term in the co-ordinates and in the J-integral. It can also be seen that including the γ_{13} term in (5.9) would not give the ϵ^4 terms correctly, since this would not include the non-linear interaction between the elastic and plastic regions represented by the γ_{33} term.

Before discussing the behaviour near the crack tip it is convenient to restrict attention to materials of the power law hardening type. In these $\tau(\gamma)$ is given by

$$\begin{aligned}\tau(\gamma) &= k\left(\frac{\gamma}{\gamma_0}\right), \quad \gamma \le \gamma_0 \\ &= k\left(\frac{\gamma}{\gamma_0}\right)^N, \quad \gamma > \gamma_0,\end{aligned} \tag{5.10}$$

N being referred to as the hardening exponent. When (5.10) applies (3.8) and (3.9) are homogeneous and can be solved, Rice[4], to give

$$f_k(\gamma) = \left(\frac{\gamma}{\gamma_0}\right)^{-\mu_k} \quad \text{and} \quad f_k^* = \left(\frac{\gamma}{\gamma_0}\right)^{-\mu_k^*} \tag{5.11}$$

with

$$\mu_k = \left\{\frac{(1-N)^2}{4} + (2k-1)^2 N\right\}^{1/2} - \frac{(1-N)}{2}$$

and

$$\mu_k^* = \left\{\frac{(1-N)^2}{4} + 4k^2 N\right\}^{1/2} - \frac{(1-N)}{2}. \tag{5.12}$$

Substituting (5.11) into (3.13) gives

$$C_k = \frac{2k-1-\mu_k}{2k-1+\mu_k} \quad \text{and} \quad C_k^* = \frac{2k-\mu_k^*}{2k+\mu_k^*}. \tag{5.13}$$

The stress–strain law (5.10) incorporates two interesting limits. If $N = 0$ then $\mu_k = \mu_k^* = 0$, $C_k = C_k^* = 1$ and an elastic–perfectly plastic material is described. If $N = 1$ then $\mu_k = 2k - 1$, $\mu_k^* = 2k$, $C_k = C_k^* = 0$ and the linear elastic solution is recovered.

The plastic region solution in a power law hardening material can be found by substituting (5.10) and (3.7) into (3.2). There results

$$\begin{aligned}-x_1 + ix_2 = e^{-i\phi}\Bigg[&\frac{1}{\gamma}\left(\sum_{k=1}^{\infty}(2k-1)D_k\left(\frac{\gamma}{\gamma_0}\right)^{-\mu_k}\cos(2k-1)\phi - \sum_{k=1}^{\infty}2kD_k^*\left(\frac{\gamma}{\gamma_0}\right)^{-\mu_k^*}\sin 2k\phi\right) \\ &-\frac{i}{\gamma}\left(\sum_{k=1}^{\infty}-\mu_k D_k\left(\frac{\gamma}{\gamma_0}\right)^{-\mu_k}\sin(2k-1)\phi + \sum_{k=1}^{\infty}\mu_k^* D_k^*\left(\frac{\gamma}{\gamma_0}\right)^{-\mu_k^*}\cos 2k\phi\right)\Bigg].\end{aligned} \tag{5.14}$$

If the material is not perfectly plastic, i.e. $N \neq 0$, then $\mu_k < \mu_{k+1}$, $\mu^*_k < \mu^*_{k+1}$ and $\mu_1 < {}^*_1$ and so for $\gamma \gg \gamma_0$ the right side of (5.14) is dominated by the D_1 terms. This means that the physical co-ordinates very near the crack tip are

$$-x_1 + ix_2 = e^{-i\phi}\frac{D_1}{\gamma}\left(\frac{\gamma}{\gamma_0}\right)^{-N} [\cos\phi - iN\sin\phi] \tag{5.15}$$

which imply a dominant strain singularity of the form

$$\begin{Bmatrix}\gamma_{13}\\ \gamma_{23}\end{Bmatrix} = (-D_1)^{1/(N+1)}\gamma_0^{N/(N+1)}r^{-1/(N+1)}[\cos^2\phi + N^2\sin^2\phi]^{1/[2(N+1)]}\begin{Bmatrix}-\sin\phi\\ \cos\phi\end{Bmatrix}, \tag{5.16}$$

where r is the distance from the crack tip. The coefficient D_1 was found in Section 3 to be

$$D_1 = \frac{-2\gamma_0 a}{1-\gamma_0 f_1'(\gamma_0)}[\beta_1\epsilon^2 + \beta_2\epsilon^4 + \beta_3\epsilon^6] + 0(\epsilon^8) \tag{5.17}$$

with $\gamma_0 f_1'(\gamma_0) = -N$ as the material is power law hardening. Substituting for β_i from (4.1) it is seen that the only effect of asymmetric loading on crack tip conditions is a third order change in the magnitude of the strain singularity—the angular dependence being unaltered. It is of interest to note that the J integral can be calculated from the singular field (5.15); as expected the result is the same as (5.6) which was found from the outer solution. In particular it may be noted that

$$D_1 = \frac{-2J}{\pi k(1+N)} \tag{5.8}$$

to the order retained and, by implication, to all orders.

If the material is elastic–perfectly plastic the above results do not apply since each term of (5.14) then contributes to the dominant strain singularity. In this case the plastic region solution can be written as

$$x_1 + ix_2 = -e^{i\phi}\gamma^{-1}\left[\sum_{k=1}^{\infty}(2k-1)D_k\cos(2k-1)\phi - \sum_{k=1}^{\infty}2kD^*_k\sin 2k\phi\right] \tag{5.19}$$

which implies strains

$$\begin{Bmatrix}\gamma_{13}\\ \gamma_{23}\end{Bmatrix} = -r^{-1}\left[\sum_{k=1}^{\infty}(2k-1)D_k\cos(2k-1)\phi - \sum_{k=1}^{\infty}2kD^*_k\sin 2k\phi\right]\begin{Bmatrix}-\sin\phi\\ \cos\phi\end{Bmatrix}. \tag{5.20}$$

It can be seen from (5.19) that the polar angle in the strain plane ϕ, can be equated to the polar angle at the crack tip in the physical plane, θ. Hence (5.20) can be written as

$$\begin{Bmatrix}\gamma_{13}\\ \gamma_{23}\end{Bmatrix} = \frac{\gamma_0 R(\theta)}{\gamma}\begin{Bmatrix}-\sin\theta\\ \cos\theta\end{Bmatrix} \tag{5.21}$$

where

$$R(\theta) = -\frac{1}{\gamma_0}\left[\sum_{k=1}^{\infty}(2k-1)D_k\cos(2k-1)\theta - \sum_{k=1}^{\infty}2kD^*_k\sin 2k\theta\right] \tag{5.22}$$

is the distance from the crack tip to the elastic–plastic boundary at an angle θ. The maximum value of $R(\theta)$ is given by (5.2) as

$$R(\theta)_{\max} = a[2\gamma^2_{11}\epsilon^2 + (6\gamma_{13}\gamma^3_{11} - 4\gamma^2_{12}\gamma^2_{11} + 2\gamma_{33}\gamma^4_{11})\epsilon^4 + 0(\epsilon^6)] \tag{5.22}$$

occurring when

$$\theta = -4\gamma_{12}\epsilon + 0(\epsilon^2), \tag{5.23}$$

and hence the strain singularity implied by (5.21) is not symmetric about the crack plane. It may be noted that the expression $\gamma_{12}\epsilon$ appearing in (5.23) can be found from the linear elastic solution of the problem. It is simply the normalised stress parallel to the crack acting at the tip.

6. A CRACK AT AN ANGLE TO A UNIFORM FIELD

The preceding results are now applied to a crack of length $2a$ perturbing a uniform stress field in a power law hardening material, as in Fig. 3. The functions $g_i'(z)$ defined in Section 2 can be

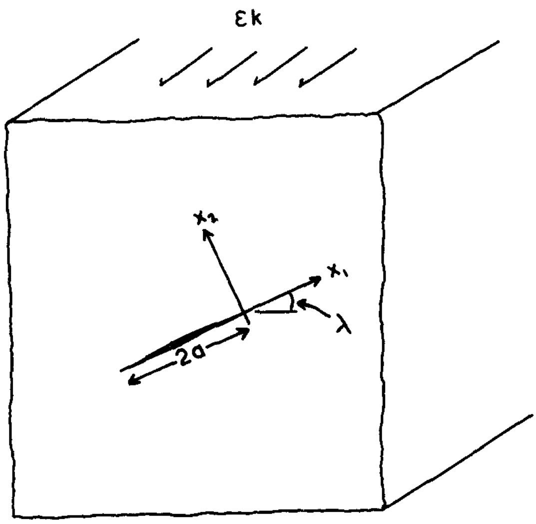

Fig. 3. A crack at an angle to a uniform stress field.

found by inspection to be

$$\begin{aligned}
g_1'(z) &= (z+1)(z^2+2z)^{-1/2}\cos\lambda + i\sin\lambda \\
g_2'(z) &= 2(z+1)(z^2+2z)^{-1} \\
g_3'(z) &= 2\sqrt{2}(z+1)(z^2+2z)^{-3/2} \\
g_4'(z) &= 4(z+1)(z^2+2z)^{-2} \\
g_5'(z) &= 4\sqrt{2}(z+1)(z^2+2z)^{-5/2} + \frac{1}{\sqrt{2}}(z+1)(z^2+2z)^{-3/2}
\end{aligned} \tag{6.1}$$

where λ is the angle between the crack and the plane on which the remote shearing stresses are maximum. The functions (6.1) are expanded and compared with (2.4) to give

$$\begin{array}{lllll}
\gamma_{11} = \frac{1}{\sqrt{2}}\cos\lambda & \gamma_{12} = \sin\lambda & \gamma_{13} = \frac{3}{4\sqrt{2}}\cos\lambda & \gamma_{14} = 0 & \gamma_{15} = \frac{-5}{32\sqrt{2}}\cos\lambda \\
 & \gamma_{22} = 0 & \gamma_{23} = \frac{1}{2} & \gamma_{24} = 0 & \gamma_{25} = -\frac{1}{4} \\
 & & \gamma_{33} = \frac{1}{4} & \gamma_{34} = 0 & \gamma_{35} = -\frac{9}{32} \\
 & & & \gamma_{44} = 0 & \gamma_{45} = -\frac{1}{4} \\
 & & & & \gamma_{55} = -\frac{3}{32}.
\end{array} \tag{6.2}$$

Assembling the various results it can be seen that once the hardening exponent N, the crack orientation λ and the remote stress $k\epsilon$ are known the solution is determined.

The shape of the plastic zone for various values of N, λ and ϵ is shown in Figs. 4 and 5. These

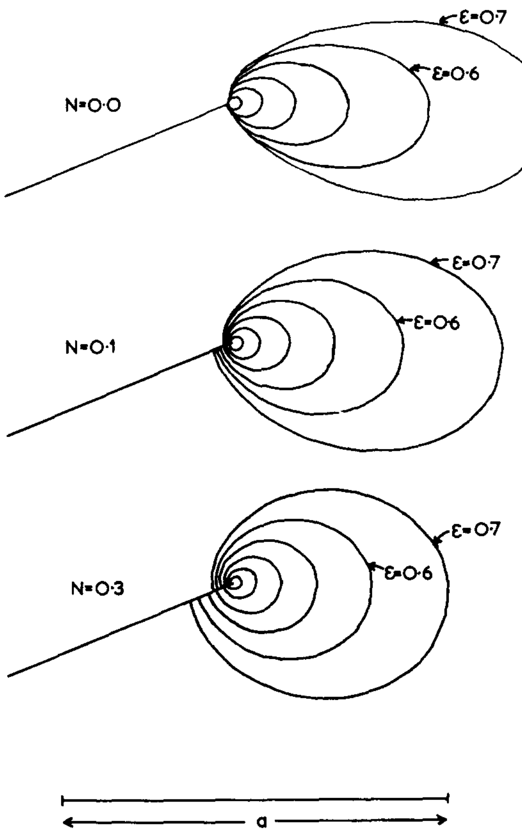

Fig. 4. Plastic zone shapes for various loadings and hardening exponents when $\lambda = \pi/8$.

are obtained from eqn (5.1) with the lengthy terms in ϵ^5 and ϵ^6 included. It is anticipated that, as in [1] and [2], the asymptotic expressions provide reasonable approximations when load levels are less than 75% of those needed for general yielding, which in this case occurs when $\epsilon = 1$ along a slip band at an angle of $-\lambda$ to the x_1 axis. The onset of breakdown of the approximation can be seen in the irregular behaviour of the elastic–plastic boundary just above the crack tip.

This effect is further illustrated in Fig. 6 where a sequence of approximations for the $N = 0$, $\lambda = \pi/4$ case is shown. The small-scale yielding approximation ($n = 1$) displays no asymmetry and is noticeably in error beyond $\epsilon \sim 0.3$. The modified boundary layer estimate ($n = 2$) is obtained from the first two terms of eqn (5.1). It shows asymmetry, but develops unrealistic kinks beyond $\epsilon \sim 0.5$. The higher order approximations ($n = 3, 4, 5$) successively defer this unlikely feature to higher values of ϵ, the approximation $n = 5$ showing evidence of breakdown at $\epsilon \sim 0.7$. (The smooth behaviour of the approximation for $n = 4$ is fortuitous and does not occur for other angles λ).

If the crack is parallel to the plane of maximum shear, i.e. $\lambda = 0$, the results may be compared with those of Amazigo [7] who gave a fully plastic solution to the problem. In the current notation Amazigo showed that, for the constitutive law

$$\frac{\tau}{k} = \left(\frac{\gamma}{\gamma_0}\right)^N \quad \text{for all } \gamma, \tag{6.3}$$

the physical co-ordinates very near the crack tip could be written as

$$x_1 - ix_2 = e^{-i\phi} \frac{2J}{\pi a k(1+N)} \left(\frac{\gamma}{\gamma_0}\right)^{-N} [\cos\phi - iN\sin\phi] \tag{6.4}$$

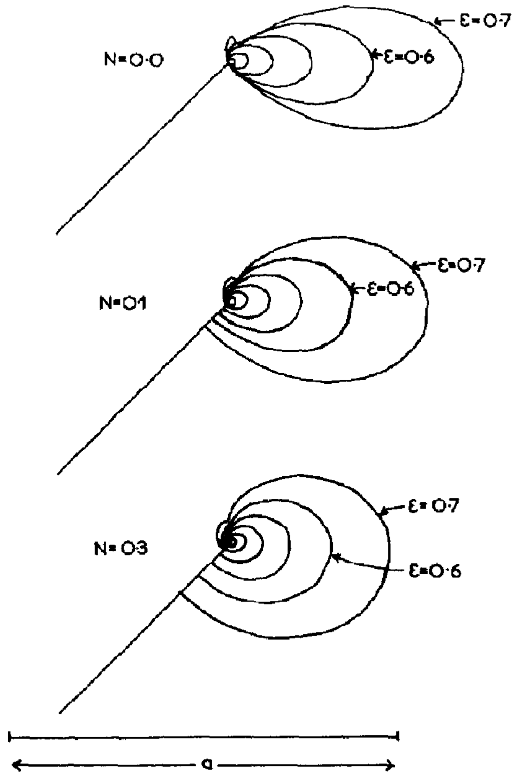

Fig. 5. Plastic zone shapes for various loadings and hardening exponents when $\lambda = \pi/4$.

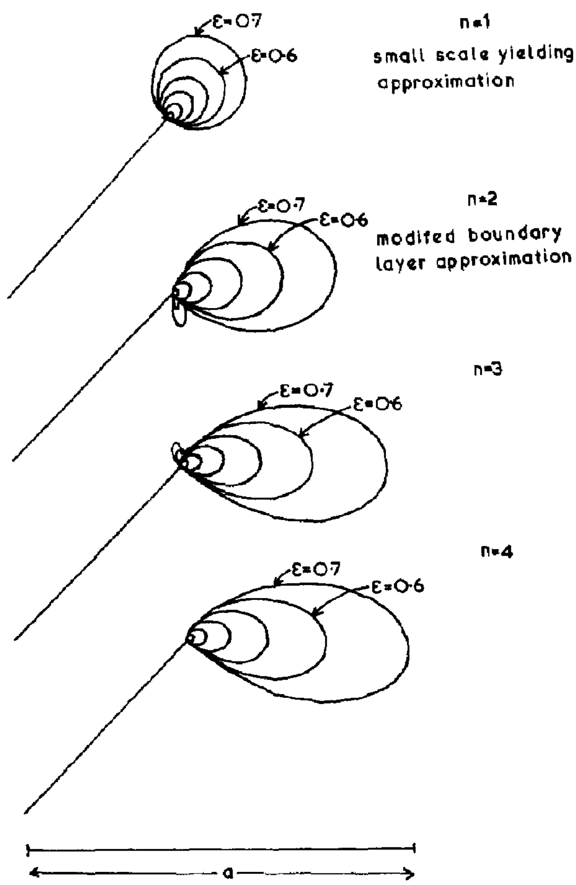

Fig. 6. Plastic zone shapes, for no hardening and $\lambda = \pi/4$, when different numbers of terms are included in the sum.

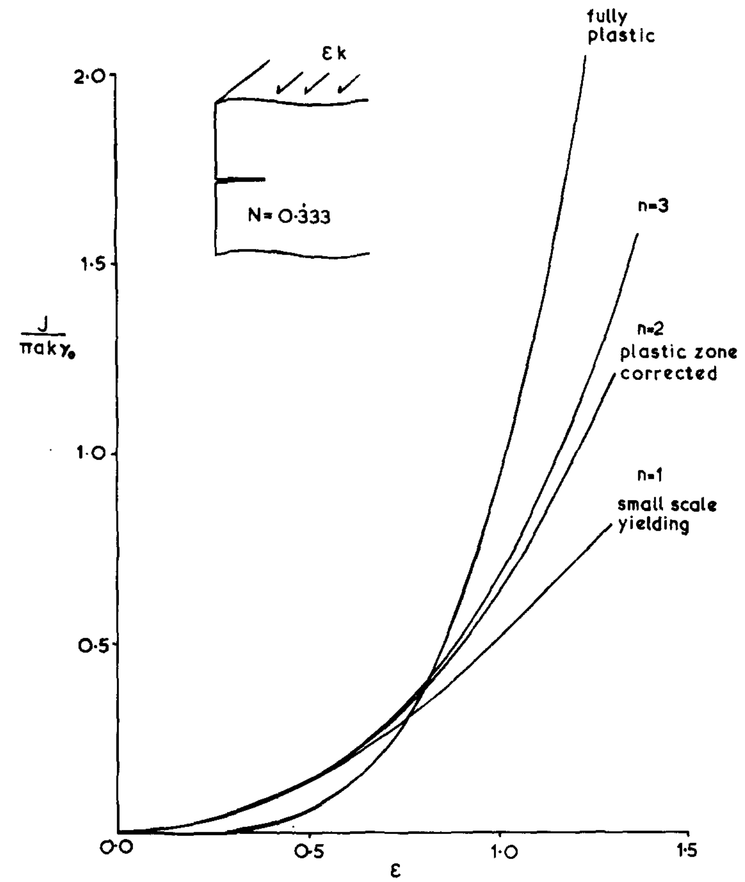

Fig. 7. The variation of J with applied load for an edge cracked plane of highly hardening material ($N = 0.333$).

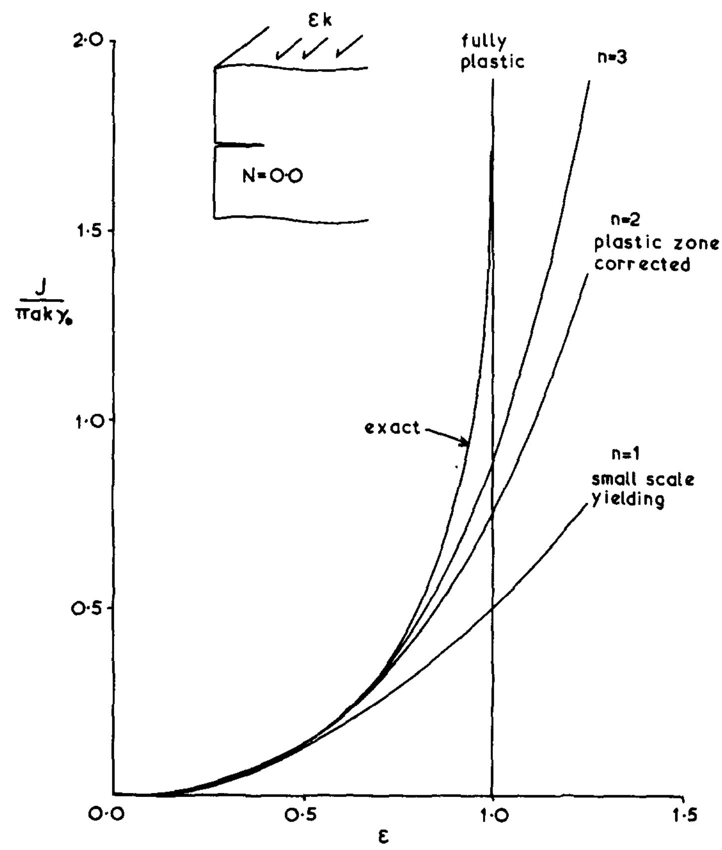

Fig. 8. The variation of J with applied load for an edge cracked plane of a non-hardening material ($N = 0$).

with $J/a\tau_\infty\gamma_\infty$ a known function of N. The equivalent result in the elastic–plastic case can be found by substituting (5.18) into (5.15) to give an expression identical to (6.4). This means that the only differences in crack tip conditions arise from the coefficient of the singularity, J. The two expressions available for J are approximations to an exact elastic–plastic result with different ranges of validity. Amazigo's fully plastic result is a good approximation when the yielding is fully developed, and the asymptotic result, (5.6), is a good approximation when yielding is confined to a small area near the crack tip. The two results are displayed in Figs. 7 and 8 where J is plotted against the remote loading for two values of the hardening exponent.

In the perfectly plastic case shown in Fig. 8, the fully plastic solution of Amazigo degenerates into the vertical asymptote at $\epsilon = 1.0$. It has also been possible to include an exact elastic perfectly plastic result, obtained from Rice[8]. These figures show that the small scale yielding approximation begins to break down at ~40% of general yielding, the plastic zone corrected result at ~60% and, in the $N = 0$ case, the three term result at ~75%. It may also be noted that any reasonable interpolation scheme between the asymptotic result and the fully plastic result would lead to small errors occurring over a restricted load range.

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